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The Tune Shift Due to Linear Coupling

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1. Introduction

In RHIC the coupling between X and Y degrees of freedom is expected from various sources.[1,2] We shall specifically examine the machine tune shift produced by skew-quadrupoles randomly distributed around the ring. In this case the X-Y coupling is linear, and may be calculated exactly within a model in which skew-quadrupole magnets are treated as point objects of strengths q_k and locations s_k, $k = 1, \ldots, N$[3]. This approximation is justified by comparing the length $\ell = 0.6$ m of a quadrupole magnet and the length $C = 3833.852$ m of RHIC's circumference, $\ell/C \sim 10^{-4}$.

A transfer matrix $\overset{\circ}{T}_{SQ}$ of a single skew-quadrupole magnet of length ℓ in the thin lense approximation (in the circular representation denoted with $\circ$ above T_{SQ}) is given by

$$\overset{\circ}{T}_{SQ} = BT_{SQ}B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{bmatrix}, \tag{1.1}$$

where

$$q = (\beta_x \beta_y)^{1/2} \frac{\ell}{\rho} a_1, \tag{1.2}$$

represents the strength of a skew-quadrupole field, and B contains lattice functions of a perfect ring

$$B = \begin{bmatrix} B_x & 0 \\ 0 & B_y \end{bmatrix}, \tag{1.3}$$

$$B = \begin{bmatrix} \beta_x^{-1/2} & 0 \\ \alpha_x \beta_x^{-1/2} & \beta_x^{1/2} \end{bmatrix}, \quad \text{similar for } B_y\ . \tag{1.4}$$

The perfect lattice has the following transfer matrix:

$$\overset{\circ}{T}^{(0)} = \begin{bmatrix} R(\mu_x) & 0 \\ 0 & R(\mu_y) \end{bmatrix}, \tag{1.5}$$

where $R(\mu_x)$ and $R(\mu_y)$ are rotations, and μ_x, μ_y are tunes ($\mu = 2\pi\nu$),

$$R(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}. \tag{1.6}$$

For RHIC, we have $\nu_x = 28.826$ and $\nu_y = 28.821$.

2. Calculation of the Tune Shift

The full transfer matrix can be written as a polynominal in the q's

$$\overset{\circ}{T} = \begin{bmatrix} \overset{\circ}{M} & \overset{\circ}{n} \\ \overset{\circ}{m} & \overset{\circ}{N} \end{bmatrix} = \sum_{k=0}^{N} \overset{\circ}{T}^{(k)}, \tag{2.1}$$

where submatrices $\overset{\circ}{M}^{(k)}$, $\overset{\circ}{n}^{(k)}$ etc. are given by k-th order in the q's driving terms. For the purpose of this note it will be sufficient to display $\overset{\circ}{M}$, $\overset{\circ}{N}$ submatrices up to the second order in the q's only.

$$\overset{\circ}{M} = R(\mu_x) + \overset{\circ}{M}^{(2)} + \text{higher terms of even order}, \tag{2.2}$$

where

$$\begin{aligned} \overset{\circ}{M}^{(2)} = \frac{1}{4}\sum_{r<s} q_r q_s \Big\{ & R\left(\mu_x + \mu_x^r + \mu_y^r - \mu_x^s - \mu_y^s\right) - R\left(\mu_x + \mu_x^r - \mu_y^r - \mu_x^s + \mu_y^s\right) + \\ & + \left[R\left(\mu_x - \mu_x^r + \mu_y^r - \mu_x^s - \mu_y^s\right) - R\left(\mu_x - \mu_x^r - \mu_y^r - \mu_x^s + \mu_y^s\right)\right] J \Big\}, \end{aligned} \tag{2.3}$$

and J is one of the fundamental Pauli matrices

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.4}$$

and μ_x^r, μ_y^r denote phase advances between the point of observations $s = 0$ and the location of s_r of the r-th skew-quadrupole

$$\mu_x^r = \int_0^{s_r} \frac{ds}{\beta_x}, \text{ and similar for } \mu_y^r. \tag{2.5}$$

Using symmetry arguments a corresponding expansion for the $\overset{\circ}{N}$ submatrix is obtained as

$$\overset{\circ}{N} = \overset{\circ}{M} |_{x \leftrightarrow y}. \tag{2.6}$$

The presence of skew-quadrupole fields produces the differences

$$\begin{aligned} &\frac{1}{2}Tr\ \overset{\circ}{M} - \cos\mu_x \neq 0, \\ &\frac{1}{2}Tr\ \overset{\circ}{N} - \cos\mu_y \neq 0, \end{aligned} \tag{2.7}$$

where this means that actual machine tunes are shifted relative to the tunes of a perfect machine. Substituting relevant traces of $\overset{\circ}{M}$ and $\overset{\circ}{N}$ one finds the final expression for the tune shifts in terms of the second order driving terms,

$$\begin{aligned} \Delta\mu_x &= -\frac{1}{2}\sum_{r<s} q_r q_s \cos\left(\mu_x^s - \mu_x^r\right)\sin\left(\mu_y^s - \mu_y^r\right) + \\ &= \frac{1}{2}\cot\mu_x \sum_{r<s} q_r q_s \sin\left(\mu_x^s - \mu_x^r\right)\sin\left(\mu_y^s - \mu_y^r\right) + 0\left(q^4\right), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \Delta\mu_y &= -\frac{1}{2}\sum_{r<s} q_r q_s \sin\left(\mu_x^s - \mu_x^r\right)\cos\left(\mu_y^s - \mu_y^r\right) + \\ &+ \frac{1}{2}\cot\mu_y \sum_{r<s} q_r q_s \sin\left(\mu_x^s - \mu_x^r\right)\sin\left(\mu_y^s - \mu_y^r\right) + 0\left(q^4\right) . \end{aligned} \tag{2.9}$$

The second order driving terms are defined as follows

$$\begin{bmatrix} d_{ss}^{(2)} \\ d_{sc}^{(2)} \\ d_{cs}^{(2)} \\ d_{cc}^{(2)} \end{bmatrix} = \sum_{1\leq r<s\leq N} q_r q_s \sin\left(\mu_y^s - \mu_y^r\right) \begin{bmatrix} \sin\mu_x^s & \sin\mu_x^r \\ \sin\mu_x^s & \cos\mu_x^r \\ \cos\mu_x^s & \sin\mu_x^r \\ \cos\mu_x^s & \cos\mu_x^r \end{bmatrix} . \tag{2.10}$$

Additional sets of the second order driving terms denoted $\check{d}_{ss}^{(2)}$, $\check{d}_{sc}^{(2)}$ etc. are obtained from the above definitions by simply exchanging x and y.

Let us notice that the tune shift vanishes when the tune splitting is corrected. This is most easily seen by first writing $\Delta\mu_x$ and $\Delta\mu_y$ as

$$\Delta\mu_x = -\frac{1}{2}\left(d_{cc}^{(2)} + d_{ss}^{(2)}\right) - \frac{1}{2}\left(d_{cs}^{(2)} - d_{sc}^{(2)}\right)\cot\mu_x + 0\left(q^4\right), \tag{2.11}$$

$$\Delta\mu_y = -\frac{1}{2}\left(\check{d}_{cc}^{(2)} + \check{d}_{ss}^{(2)}\right) - \frac{1}{2}\left(\check{d}_{cs}^{(2)} - \check{d}_{sc}^{(2)}\right)\cot\mu_y + 0\left(q^4\right). \tag{2.12}$$

On correction of the tune splitting one requires that the following conditions hold

$$\begin{aligned} d_{sc}^{(2)} - d_{cs}^{(2)} &= 0, \\ \check{d}_{sc}^{(2)} - \check{d}_{cs}^{(2)} &= 0, \\ d_{cc}^{(2)} + d_{ss}^{(2)} &= 0, \\ \check{d}_{cc}^{(2)} + \check{d}_{ss}^{(2)} &= 0. \end{aligned} \tag{2.13}$$

Clearly, the tune shifts $\Delta\mu_x$ and $\Delta\mu_y$ given by Eqs. (2.11) and (2.12) vanish under these conditions. This conclusion was recently obtained, using different methods by G. Parzen.[4]

3. References

1. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).
2. D.A. Edwards and L.C. Teng, IEEE Trans. Nucl. Sci. Vol. NS-20, No. 3, p. 885 (1973).
3. A.G. Ruggiero, "The Problem of Linear Coupling, Parts I, II, III", Talks at Accelerator Physics Division Meetings, May 16, 23 and 30 (1991), BNL.
4. G. Parzen, BNL Report, AD/RHIC-100 (July 1991).